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UNIFORM CONVEXITY PROPERTIES OF NORMS ON A SUPER-REFLEXIVE BANACH SPACE

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ABSTRACT

We study uniform convexity and smoothness properties satisfied by all the equivalent norms of a super-reflexive Banach space. We give some applications concerning quasi-transitive Banach spaces, and the uniform approximation property.

Introduction

P. Enflo proved that there exists on every super-reflexive Banach space an equivalent norm which is uniformly convex ([5]). This result had been made more precise by G. Pisier ([14]): every super-reflexive Banach space has an uniformly convex equivalent norm with a modulus of convexity of power-type.

A natural question is: what can be said of any equivalent norm on a super-reflexive Banach space? We show that every equivalent norm has some uniform convexity and smoothness properties.

Let us consider the notion of strong extreme point (this notion has been introduced by K. Kunen and H. Rosenthal ([11])). We can define a modulus of strong extremality as follows: let x be a point of the unit sphere; the modulus of strong extremality in x is the number:

$$\forall \varepsilon > 0, \qquad \Delta(x, \varepsilon) = \inf\{1 - \lambda\,;\, \exists \tau : \|\lambda x \pm \tau\| \leqq 1, \|\tau\| \geqq \varepsilon\}.$$

It is easy to show that a point x of the unit sphere is a strong extreme point of the unit ball if and only if $\Delta(x, \varepsilon) > 0$, $\forall \varepsilon > 0$.

In Section I, we will show that if E is a super-reflexive Banach space and B is the unit ball of a given equivalent norm on E, then B has "many" strong

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extreme points with an uniform minoration of the modulus of strong extremality of these points.

Our major tool will be an adaptation of a technique of J. Lindenstrauss ([12]).

In Section II, we will give some applications: duality with smoothness properties, quasi-transitive Banach spaces and uniform approximation property.

Notations

Let X be a Banach space and N be a norm on X. We denote $B_N(X)$ (or $B(X)$) the unit ball of X, $S_N(X)$ the unit sphere and X^* its dual. If F is a subset of X, $\operatorname{conv} F$ is the convex hull of F.

I. Strong extreme points

We will show that every unit ball in a super-reflexive Banach space is "up to η" contained in the convex hull of a subset of its strong extreme points which satisfies a condition of uniformity.

DEFINITION 1 ([11]). Let C be a closed convex bounded set. A point x in C is a strong extreme point if for every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that

$$y, z \in C, \qquad \left\| \frac{y+z}{2} - x \right\| \leqq \eta(\varepsilon) \Rightarrow \| y - z \| \leqq \varepsilon.$$

Let us compare this notion with classical ones ([4]). The following can be shown:

(1) x strongly exposed $\underset{\Rightarrow}{\not\Leftarrow}$ x strong extreme $\underset{\Rightarrow}{\not\Leftarrow}$ x extreme.

(2) A norm is locally uniformly convex $\underset{\Rightarrow}{\not\Leftarrow}$ every point of the unit sphere is a strong extreme point of the unit ball $\underset{\Rightarrow}{\not\Leftarrow}$ the norm is strictly convex (see [4]).

The modulus $\Delta(x, \varepsilon)$ which is defined below measures "how much" a point is a strong extreme point of the unit ball.

DEFINITION 2. Let X be a Banach space with norm $\|\cdot\|$. The modulus of strong extremality in x is the number

$$\forall \varepsilon > 0, \qquad \Delta_{\|\cdot\|}(x, \varepsilon) = \inf\{1 - \lambda\,;\, \exists \tau \, \|\lambda x \pm \tau\| \leqq 1,\, \|\tau\| \geqq \varepsilon\}.$$

REMARKS. (1) The modulus of strong extremality in a point is an increasing function and $\Delta_{\|\cdot\|}(x, 0) = 0$.

(2) It is easy to show that x is a strong extreme point of the unit ball if and only if $\Delta_{\|\cdot\|}(x, \varepsilon) > 0$, $\forall \varepsilon > 0$.

(3) Recall that the modulus of convexity (see [4]) is given by

$$\delta_{\|\cdot\|}(\varepsilon)=\inf\left\{1-\left\|\frac{x+y}{2}\right\|,\ \|x\|\leqq 1,\ \|y\|\leqq 1,\ \|x-y\|\geqq\varepsilon\right\}\quad(\varepsilon>0).$$

The modulus of convexity is of power-type if there exist $C>0$, $q\geqq 2$ such that for every $\varepsilon>0$, $\delta_{\|\cdot\|}(\varepsilon)\geqq C\varepsilon^q$. It can be proved that $\Delta_{\|\cdot\|}(x,\varepsilon)>\delta_{\|\cdot\|}(2\varepsilon)$ and conversely, if $\inf\{\Delta_{\|\cdot\|}(x,\varepsilon),\|x\|=1\}>0$, $\forall\varepsilon>0$ then the norm $\|\cdot\|$ is uniformly convex.

Let us give now the main result of this section. For any equivalent norm $\|\cdot\|$ on a super-reflexive Banach space X we let

$$\Omega_{\|\cdot\|}(K,q)=\{x\in S_{\|\cdot\|}(X):\Delta_{\|\cdot\|}(x,\varepsilon)\geqq K\varepsilon^q,\ \forall\varepsilon>0\}\quad(K>0,\ q\geqq 2).$$

With this notation, the following is true:

THEOREM 3. *Let X be a super-reflexive Banach space, and $\|\cdot\|$ be an equivalent norm on X with modulus of convexity of power-type.*

N is any equivalent norm on X. Then:

$$\forall\eta,\qquad 0<\eta<1,\ \exists K(\eta)>0$$

such that

$$B_N(X)\subseteq\operatorname{conv}[\Omega_N(K(\eta),q)]+\eta B_N(X).$$

PROOF. We begin with a lemma.

LEMMA 4. *Let $(Y,|||\cdot|||)$ be a uniformly convex space with modulus of convexity $\delta_{|||\cdot|||}$. Let $S:(X,N)\to Y$ be an isomorphism between X and a subspace $S(X)$ of Y. If S attains its norm in x, then x is a strong extreme point of $B_N(X)$ and moreover*

$$\Delta_N(x,\varepsilon)\geqq\delta_{|||\cdot|||}\left(\frac{2\varepsilon}{\|S\|\,\|S^{-1}\|}\right).$$

PROOF OF THE LEMMA. We consider $\varepsilon>0$, $k_1=\lambda x+\tau$, $k_2=\lambda x-\tau$ in the unit ball $B_N(X)$ and $N(\tau)\geqq\varepsilon$. We let $S_0=S/\|S\|$. One has

$$\|S_0^{-1}\|\ |||S_0(k_1-k_2)|||\geqq N(k_1-k_2)\geqq 2\varepsilon$$

and thus

$$|||S_0(k_1-k_2)|||\geqq\frac{2\varepsilon}{\|S_0^{-1}\|}.$$

Clearly S_0k_1 and S_0k_2 are in the unit ball of Y. Therefore,

$$\left|\left|\left|\frac{S_0(k_1+k_2)}{2}\right|\right|\right| \leqq 1-\delta_{|||\cdot|||}\left(\frac{2\varepsilon}{\|S_0^{-1}\|}\right).$$

But we have

$$\left|\left|\left|\frac{S_0(k_1+k_2)}{2}\right|\right|\right| = |||\, S_0(\lambda x)\,||| = |\lambda|.$$

Therefore

$$\lambda \leqq 1-\delta_{|||\cdot|||}\left(\frac{2\varepsilon}{\|S_0^{-1}\|}\right).$$

This means

$$\delta_{|||\cdot|||}\left(\frac{2\varepsilon}{\|S_0^{-1}\|}\right)\leqq 1-\lambda$$

and thus, by the definition of the modulus of strong extremality,

$$\delta_{|||\cdot|||}\left(\frac{2\varepsilon}{\|S_0^{-1}\|}\right)\leqq \Delta_N(x,\varepsilon).$$

Since $\|S_0^{-1}\| = \|S\|\,\|S^{-1}\|$, this shows the lemma. ■

Let us come back to the proof of the theorem. The modulus of convexity of the norm $\|\cdot\|$ of X is of power-type, thus there exist $C>0$, $q\geqq 2$ such that for every $\varepsilon>0$, $\delta_{\|\cdot\|}(\varepsilon)\geqq C\cdot\varepsilon^q$.

We let $Y = X\oplus_{l^2}\mathbf{R}$ with the norm

$$|||\,(x,r)\,||| = (\|x\|^2+r^2)^{1/2} \qquad \text{for } x\in X \text{ and } r\in\mathbf{R}.$$

An easy computation shows that

$$\delta_{|||\cdot|||}(\varepsilon)\geqq C\varepsilon^q \qquad \forall\varepsilon>0.$$

Let A, B be such that, for every $x\in X$, $AN(x)\leqq\|x\|\leqq BN(x)$, and let η be a given number, $0<\eta<1$. We have to find $K>0$ so that $B_N(X)\subseteq \operatorname{conv}[\Omega_N(K,q)]+\eta B_N(X)$.

If $B_N(X)\not\subseteq \operatorname{conv}[\Omega_N(K,q)]+\eta B_N(X)$, there exists $x\in B_N(X)$ and $x\notin \operatorname{conv}[\Omega_N(K,q)]+\eta B_N(X)$. This implies that

$$d\left(x, \operatorname{conv}[\Omega_N(K,q)]+\frac{\eta}{2}B_N(X)\right)>\frac{\eta}{4}.$$

Thus there exists $f \in X^*$, $\|f\| = 1$ such that for every $y \in \mathrm{conv}[\Omega_N(K,q)]$, $|f(y)| \leqq 1 - \eta/2$. Therefore, for every $y \in \mathrm{conv}[\Omega_N(K,q)] + \frac{1}{4}\eta B_N(X)$, $|f(y)| \leqq 1 - \eta/4$.

We define an operator T from X to Y by $Tx = (x, \lambda f(x))$ with $\lambda = 9B/\eta$; note that λ does not depend on K. T is an isomorphism into.

We have

$$\|T\|^2 = \sup\{\|x\|^2 + \lambda^2 |f(x)|^2, N(x) \leqq 1\}$$

On the other hand, one has for $x \in \mathrm{conv}[\Omega_N(K,q)] + \frac{1}{4}\eta B_N(X)$

$$|||\, Tx \,|||^2 \leqq B^2 + \lambda^2 \left(1 - \frac{\eta}{4}\right)^2 \leqq (\lambda - B)^2.$$

Since X is reflexive, there exists an isomorphism S from X into Y attaining its norm in a point t in $B_N(X)$ and close to T ([12]). Let S and t be such that

$$\|S - T\| \leqq B\frac{\eta}{16} \quad \text{and} \quad |||\, St \,||| = \|S\|.$$

One has

$$|||\, St \,||| \geqq \|T\| - \frac{B\eta}{16} \geqq B\left(\frac{9}{\eta} - \frac{\eta}{16}\right). \tag{1}$$

On the other hand, if $x \in \mathrm{conv}[\Omega_N(K,q)] + \frac{1}{4}\eta B_N(X)$:

$$|||\, Sx \,||| \leqq \sup\left\{ |||\,(T-S)(y)\,|||,\ y \in \mathrm{conv}[\Omega_N(K,q)] + \frac{\eta}{4} B_N(X)\right\} + |||\, T(x) \,|||$$

$$\leqq B\frac{\eta}{16}\left(1 + \frac{\eta}{4}\right) + B\left(\frac{9}{\eta} - 1\right). \tag{2}$$

From the inequalities (1), (2), we deduce that $t \notin \mathrm{conv}[\Omega_N(K,q)] + \frac{1}{4}\eta B(X)$.

We let $S_0 = S/\|S\|$. Let us estimate the norm of S_0^{-1}. One has

$$\begin{aligned}\|T\|^2 &= \sup\{\|x\|^2 + \lambda^2 |f(x)|^2, N(x) \leqq 1\} \\ &\leqq B^2\left(1 + \frac{9}{\eta}\right)^2.\end{aligned}$$

Thus

$$\|S\| \leqq B\left(1 + \frac{9}{\eta} + \frac{\eta}{16}\right) \leqq B\frac{11}{\eta}.$$

We also have $\|T^{-1}\| \leqq 1/A$ and

$$\|S^{-1}\| \leqq \|T^{-1}\| \left(1+\frac{B\eta}{16}\right) \leqq \frac{1}{A}\left(1+\frac{B\eta}{16}\right).$$

Thus

$$\|S_0^{-1}\| = \|S\|\,\|S^{-1}\| \leqq \frac{B}{A}\frac{11}{\eta}(1+B)$$

and this norm does not depend on K.

The operator S_0 (with $\|S_0\|=1$) attains its norm in the unit ball but not on the set $\mathrm{conv}[\Omega_N(K,q)]+\frac{1}{4}\eta B(X)$; S_0 is an isomorphism into and moreover

$$\|S_0^{-1}\| \leqq \frac{B}{A}\frac{11}{\eta}(1+B).$$

Now Lemma 4 implies that the point t in which S_0 attains its norm satisfies

$$\Delta_N(t,\varepsilon) \geqq C\left(\frac{2\varepsilon}{\|S\|\,\|S^{-1}\|}\right)^q.$$

Thus we have

$$\Delta_N(t,\varepsilon) \geqq C\left(\frac{2A^2\eta}{11(1+B)B}\right)^q \varepsilon^q;$$

and K such that $B_N(X) \not\subseteq \mathrm{conv}[\Omega_N(K,q)]+\eta B_N(X)$ cannot be smaller than $K(\eta)$ such that S_0 attains its norm on $\Omega_N(K(\eta),q)$.

Therefore, if we take

$$K(\eta) = C\left(\frac{2A^2\eta}{11(1+B)B}\right)^q$$

we have

$$B_N(X) \subseteq \mathrm{conv}(\Omega_N(K(\eta),q)) + \eta B_N(X). \qquad \blacksquare$$

Remarks. (1) In the case where $\dim X$ is finite, this result can be obtained more directly by using arguments of strong compacity.

(2) It could be noted that the expression

$$K(\eta) = C\left(\frac{2A^2\eta}{11B(1+B)}\right)^q$$

is uniform on the norms N satisfying $AN(x) \leqq \|x\| \leqq BN(x)$.

(3) The example of $X = \bigoplus_{l^2} l_n^\infty$ shows that the theorem is not true in general for a reflexive space X (see Section II below). It would be nice to know if the

validity of Theorem 3 characterizes the class of super-reflexive Banach spaces in the following sense:

(*) For every equivalent norm N on X, for every $\eta > 0$, there exists $K_{N,\eta}(\cdot) > 0$ such that $B_N(X) \subseteq \operatorname{conv} \Omega_{N,\eta} + \eta B_N(X)$ where

$$\Omega_{N,\eta} = \{x \in S_N(X) : \Delta_N(x, \varepsilon) \geqq K_{N,\eta}(\varepsilon), \forall \varepsilon > 0\}.$$

Is it true that the (*) property implies the super-reflexivity?

(4) We introduce the notion of φ-strongly exposed point. In what follows we denote by φ an increasing function in $[0,1]$ such that $\varphi(0) = 0$.

Definition 5. Let C be a subset of a Banach space X and $x \in C$. We say that x is φ-strongly exposed in C if

(1) there exists $f \in X^*$ such that $f(x) = \sup\{f(y),\ y \in C\}$,

(2) if $y \in C$ satisfies $f(x) - f(y) \leqq \varphi(\varepsilon)$ for some $\varepsilon \in]0,1[$ then $\|x - y\| \leqq \varepsilon$.

Then f is called a φ-strongly exposing functional of x.

Let $\|\cdot\|$ be a norm of a Banach space X, let us denote $\mathscr{E}_{\|\cdot\|}(\varphi)$ the set of the φ-strongly exposed points in the unit ball $B_{\|\cdot\|}(X)$.

The proof of the following statement is very similar to the proof of Theorem 3. We do not give it for the sake of shortness

Proposition 6. *Let X be a super-reflexive Banach space and $\|\cdot\|$ be an uniformly convex norm on X such that $\delta_{\|\cdot\|}(\varepsilon) \geqq C\varepsilon^q$, $\forall \varepsilon > 0$; N is an equivalent norm. Then, for every $\eta \in]0,1[$ there exist a function φ_η and a constant $K(\eta)$ such that*

$$B_N(X) \subseteq \operatorname{conv}[\mathscr{E}_N(\varphi_\eta) \cap \Omega_N(K(\eta), q)] + \eta B_N(X).$$

Remark. By using an argument of J. M. Borwein ([2]) it is possible to show that the family of the φ_n-strongly exposing functionals for a point of the unit ball is an η-net in $S(X^*)$.

II. Applications

1. Duality with Smoothness Properties

We say that a Banach space $(X, \|\cdot\|)$ belongs to the *class* $\mathscr{C}$ if:
for every $\eta \in]0,1[$, there exists a function φ_η such that

$$B_{\|\cdot\|}(X) \subseteq \operatorname{conv}(\mathscr{E}_{\|\cdot\|}(\varphi_\eta)) + \eta B_{\|\cdot\|}(X).$$

When this property of uniform exposition is transformed by duality, we obtain a condition of uniform smoothness, more precisely: let us recall a definition which has been introduced in ([6], [7]).

Let X be a Banach space. $\mathscr{D}(X)$ is the set of the x (in the unit sphere) where the norm is Fréchet-smooth, and for every $x \in \mathscr{D}(X)$, we denote f_x the differential of this norm in x.

DEFINITION 7. We say that X is almost uniformly smooth (a.u.s.) if there exists a family $(A_\varepsilon)_{0<\varepsilon<1}$ of subsets of $\mathscr{D}(X)$ such that:

(a) $\forall \varepsilon \in]0,1[$, $\exists \delta(\varepsilon)>0$ such that $y \in B(X^*)$, $x \in A_\varepsilon$ and $y(x) > 1-\delta(\varepsilon) \Rightarrow \| y - f_x \| < \varepsilon$.

(b) The set $X_\varepsilon = \{f_x, x \in A_\varepsilon\}$ is a $(1-\varepsilon)$-norming subset of X^* (that is, $\forall y \in X$, $\sup\{f_z(y), z \in A_\varepsilon\} \geqq (1-\varepsilon)\| y \|$).

Let us point out that this terminology is different from the terminology used in ([6]). The duality between the class $\mathscr{C}$ and the class of almost uniformly smooth spaces is illustrated by the following proposition.

PROPOSITION 8. *X belongs to the class $\mathscr{C}$ if and only if X^* is almost uniformly smooth.*

PROOF. (1) Suppose X is in the class $\mathscr{C}$ and, for every $\varepsilon > 0$, let

$$A_\varepsilon = \{y^* \in S(X^*) : y^* \ \varphi_\varepsilon\text{-strongly exposes a point } y \text{ of } X\}.$$

If $y^* \in A_\varepsilon$, then X^* is smooth in y^* and $f_{y^*} = y$. The uniformity can be deduced by the computation which is used in the proof of V. L. Smulyan ([4], [15]).

(2) Suppose now X^* is a.u.s. and $X \notin \mathscr{C}$. There exists η such that for every φ, $B(X) \not\subseteq \operatorname{conv}(\mathscr{E}(\varphi)) + \eta B(X)$. Therefore, $\exists f$, $\|f\| = 1$ and for every $y \in \operatorname{conv}(\mathscr{E}(\varphi)) + \eta/4 B(X)$,

$$|f(y)| \leqq 1 - \eta/4. \tag{1}$$

Let $x \in A_\eta$, then $\forall y \in S(X^{**})$, $1 - y(x) \leqq \delta(\eta) \Rightarrow \| y - f_x \| \leqq \eta$; and f_x is φ_η-strongly exposed by x and $f_x \in X$. Then (1) implies that $|f(f_x)| \leqq 1 - \eta/4$, for every $x \in A_{\eta/8}$. This is impossible since $X_{\eta/8}$ is an $(1-\eta/8)$-norming subset of X^{**}. ∎

With this terminology, Propositions 6 and 8 give us the following result:

PROPOSITION 9. *Every super-reflexive space is almost uniformly smooth for every equivalent norm.*

REMARKS. (1) Let us notice that it is not true in general for reflexive spaces (an example is the space $\bigoplus_{l^2} l_n^1$, see [6]).

(2) The almost uniform smoothness property is far from implying reflexivity. Examples of a.u.s. spaces are given in [6]: $c_0(\Gamma)$, $l^\infty(\Gamma)$, $K(l^p, l^q)$, $\mathscr{L}(l^p, l^q)$ $(1 < p, q < \infty)$. If X and Y are a.u.s. and Y^* has the Radon–Nikodym property and the approximation property then the tensor-product $X \hat{\otimes}_\varepsilon Y$ is a.u.s. ([6]).

A natural motivation for the introduction of this class is the following: among the a.u.s. Banach spaces, a nice characterization of dual spaces (in the isometric sense) is available (see [6]).

THEOREM 10. *Let X be an almost uniformly smooth Banach space. Then the following assertions are equivalent*:

(1) *X is isometric to a dual space.*

(2) *For every bounded subset B of X, one has $r_X(B) = r_{X^{**}}(B)$, where $r_X(B)$ denotes the Tchebychev radius of B in X.*

2. *Quasi-transitive Banach Spaces*

Let X be a super-reflexive space. Let us assume that X is quasi-transitive in the following sense ([1]): there exists $x \in S(X)$ such that the set $\{I(x), I$ bijective isometry of $X\}$ is norm-dense in $S(X)$. Then X is uniformly convex with a modulus of convexity of power-type. Indeed, by Theorem 3, there exists at least one point x_0 such that $\Delta(x_0, \varepsilon) \geqq C\varepsilon^q$. It is easy to show that $\Omega = \{I(x_0), I$ bijective isometry$\}$ is norm-dense in $S(X)$ and one has $\Delta(y, \varepsilon) = \Delta(x_0, \varepsilon)$ for every $y \in \Omega$, and every $\varepsilon > 0$. By the norm-density of Ω one easily shows that $\Delta(y, \varepsilon) \geqq C\varepsilon^q$ for every $y \in S(X)$ and $\varepsilon > 0$, and this shows that X is uniformly convex with modulus of convexity of power-type.

EXAMPLE. $L^p([0,1], dt)$ (with $1 < p < \infty$) satisfies this property ([1]).

3. *Uniform Approximation Property*

Let us recall the definition ([13])

DEFINITION 11. A Banach space X is said to have the λ-uniform approximation property (λ-u.a.p.) if $\forall \varepsilon > 0$, $\forall k$ integer, $\forall F$ subspace of X with $\dim F = k$, *there exists an operator* $T : X \to X$ with

(1) $\operatorname{rk}(T) \leqq n_X(k, \varepsilon)$,

(2) $\|T\| \leqq \lambda$,

(3) $\|Tx - x\| \leqq \varepsilon$ for $x \in B(F)$,

where $n_X(k, \varepsilon)$ is an integer which depends on k and ε, but not on the space F.

J. Lindenstrauss and L. Tzafriri have proved that a super-reflexive space X has 1-u.a.p. if and only if X^* has 1-u.a.p. ([13]) (S. Heinrich extended this result to general spaces by using the ultrapowers ([8])). Theorem 3 permits one to get their result and an explicit computation of $n_{X^*}(k,\varepsilon)$ for every equivalent norm on X.

First, let us recall the

DEFINITION 12 ([3]). *A Banach space X has the* convex approximation property *if for every $\varepsilon > 0$, there exists an integer p such that*

$$\forall A \subset B(X), \qquad \operatorname{conv} A \subseteq \operatorname{conv}_p A + \varepsilon B(X)$$

where

$$\operatorname{conv}_p A = \left\{ \sum_{i=1}^{p} \lambda_i x_i\,;\ \lambda_i \geqq 0,\ \sum_{i=1}^{p} \lambda_i = 1,\ x_i \in A \right\}.$$

R. E. Bruck has proved that the convex approximation property and the B-convexity are equivalent ([3]).

We consider a Banach space X such that X^* is B-convex and in the class $\mathscr{C}$. Fix $\varepsilon > 0$; we note $p(\varepsilon)$ is an integer such that $\forall A \subset B(X^*)$,

$$\operatorname{conv} A \subseteq \operatorname{conv}_{p(\varepsilon)} A + \varepsilon B(X^*)$$

and φ_ε is a function such that

$$B(X^*) \subseteq \operatorname{conv} \mathscr{E}(\varphi_\varepsilon) + \varepsilon B(X^*).$$

If k is an integer and F a subspace of dimension k, the cardinal of an ε-net of the unit sphere of F is maximized by $K \cdot \varepsilon^{-k}$ where K is a constant which does not depend on F. With these notations, we get

THEOREM 13. *Let X be a reflexive Banach space with X^* B-convex and in the class $\mathscr{C}$. Suppose that X has* 1*-u.a.p., then for every $\varepsilon > 0$, k integral, one has*

$$n_{X^*}(k, 9\varepsilon) \leqq n_X(K\varepsilon^{-k} p(\varepsilon), \varphi_\varepsilon(\varepsilon)).$$

PROOF. Fix $\varepsilon > 0$. Let $F \subset X^*$, $\dim F = k$. We consider an ε-net, $y_1, \ldots, y_{l(k,\varepsilon)}$, in the unit sphere of F where $l(k,\varepsilon) \leqq K\varepsilon^{-k}$.

Let $x \in S(F)$. There exists an integer $i \in \{1, \ldots, l(k,\varepsilon)\}$ such that $\|x - y_i\| \leqq \varepsilon$; $y_i \in \operatorname{conv} \mathscr{E}(\varphi_\varepsilon) + \varepsilon B(X)$. Thus there exists $z_i \in \operatorname{conv}_{p(\varepsilon)} \mathscr{E}(\varphi_\varepsilon)$ such that $z_i = \sum_{j=1}^{p(\varepsilon)} \lambda_j z_{ij}$ with

$$\lambda_j \geqq 0, \quad \sum_{j=1}^{p(\varepsilon)} \lambda_i = 1, \quad z_{ij} \in \mathscr{E}(\varphi_\varepsilon), \quad \|y_i - z_i\| \leqq 3\varepsilon.$$

X is reflexive, thus we can find $x_{ij} \in A_\varepsilon$ such that $z_{ij} = f_{x_{ij}}$. We consider

$$F_1 = \overline{\operatorname{span}}\{x_{ij}\,;\, i = 1, \dots, l(k, \varepsilon),\, j = 1, \dots, p(\varepsilon)\},$$

$\dim F_1 \leqq l(k, \varepsilon) \cdot p(\varepsilon)$.

Let $T : X \to X$ such that $\| T \| \leqq 1$,

$$\| Ty - y \| \leqq \varphi_\varepsilon(\varepsilon), \qquad y \in B(F_1),$$

$$\operatorname{rk}(T) \leqq n_X(l(k, \varepsilon) \cdot p(\varepsilon), \varphi_\varepsilon(\varepsilon)).$$

Thus, if T^* is the conjugate of T,

$$\begin{aligned} \| T^* x - x \| &\leqq \| T^*(x - y_i)\| + \| T^* y_i - y_i \| + \| y_i - x \| \\ &\leqq 2\varepsilon + \| T^*(y_i - z_i)\| + \| T^* z_i - z_i \| + \| z_i - y_i \| \\ &\leqq 8\varepsilon + \sum_{j=1}^{p(\varepsilon)} \lambda_j \| T^* z_{ij} - z_{ij} \| \\ &\leqq 9\varepsilon. \end{aligned}$$

Indeed,

$$\begin{aligned} T^* f_{x_{ij}}(x_{ij}) = f_{x_{ij}}(Tx_{ij}) &= 1 - (f_{x_{ij}}(x_{ij}) - f_{x_{ij}}(Tx_{ij})) \\ &\geqq 1 - \| x_{ij} - Tx_{ij} \| \\ &\geqq 1 - \varphi_\varepsilon(\varepsilon). \end{aligned}$$

X satisfies the condition (a) of Definition 7, then

$$\| T^* f_{x_{ij}} - f_{x_{ij}} \| \leqq \varepsilon. \qquad \blacksquare$$

REMARK. A super-reflexive space is B-convex and in the class $\mathscr{C}$ for every equivalent norm. Moreover, if X is super-reflexive and has λ-u.a.p. then X has the 1-u.a.p. for every equivalent norm ([13]). Thus every super-reflexive space with λ-u.a.p. satisfies Theorem 13 for every equivalent norm.

Let us also note that the function φ_ε is uniform on the norms N satisfying $AN(x) \leqq \| x \| \leqq BN(x)$.

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